Maxwell's Equations 101:[1]

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Introduction:

Okay, seriously? So you just opened a document entitled “Maxwell's Equations” because, what, you were *bored* and you wanted a challenge? Well, you're a plucky little thing, aren't you. And so now you're thinking “Uh huh; how tough can this possibly be? I took *algebra* in Junior High… not to mention being a *ninth* level sword apprentice in the realm of 'Dragon Masters'. So what's the big deal?”

Seriously? What, other than the fact that Maxwell's Equations just define *one quarter of the known universe*, and are said to contain more information than the *total composite knowledge of mankind* ? Sure, these equations should be *cake* for you, what with your grand mastery of *algebra,* and sword apprenticeship, and all.

You still here? So what, now you're intrigued, or is it just your stubborn streak determined to not let the *annoying* geek at the other end of this keyboard send you *crying* back home to your *mother* (*again*). Well, now we are impressed.

Okay fine. So take a deep breath, and we'll walk through this one step at a time, and we'll just see who's the better man here. Through it all, maybe we'll even manage to expand your mind a little (just past the edge of your *grand* algebraic wizardry and all). Assuming of course you can keep from tripping over your skirt, that is.

Basic pre-flight ground roll:

Before we plunge into the deep end here, let's first talk a little about the various bits of “shorthand” we're going to be using. As with any topic or field of study, “80%” of the challenge is just “learning the lingo”, or in this case the symbols used to express certain concepts or represent certain “operations” to be performed.

If you think about it, most of the symbols we use every day have no intrinsic meaning of their own (which includes of course, all the letters on this page). However after a little time under the tutelage of your grade school/primary school *drill sergeant* of a teacher, you were slowly conditioned to automatically associate such symbols with certain specific meanings. For example:

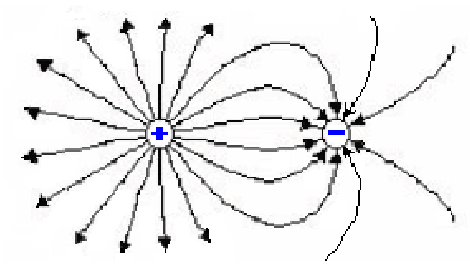
have no intrinsic underlying meaning embedded in the fabric of the universe or otherwise imprinted on our DNA. Yet when familiar symbols such as these cross our gaze, they automatically invoke an almost reflex response in us, simply because we learned over the years to associate their compact symbology with the specific meaning they are intended to represent.

One of the biggest barriers most people have in fact when encountering a new topic (e.g. the shorthand notation used in a chemical, or algebraic equation, etc.), is an almost *gut*

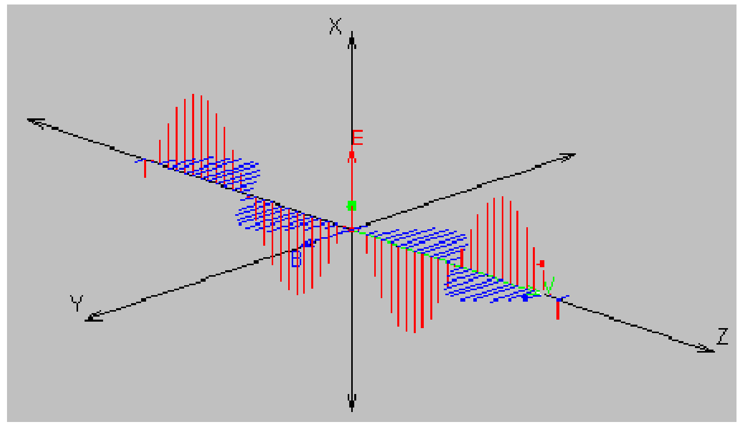
1 See “EpiphanyBySteveLee.com”, misc. for additional tutorials and projects to be added over time.

level reflex akin to *shock* at seeing notation that is an almost alien thing to them. Yet after they learn the meaning that is associated with those symbols, they can't remember what all the crying was about.

For example, " **E** (x,y,z, t) " doesn't exactly roll off the tongue. However once we learn this is a shorthand notation for how something we call the "Electric field" flows out into space, this symbology starts to develop a bit more meaning for us:



The static Electric field around 2 charges.



The sinusodial flow of an electric (E) and a magnetic (B) field along the 'z' axis.

Similarly, " **B** (x,y,z, t) " represents the magnetic field. Any time electric charges are put in motion (i.e. we develop an electric current), we then detect a Magnetic Field around those moving charges. Since the humble little electric charge, "q", is the source of both types of fields, it should come as no surprise that the Magnetic Field is directly related to the Electric field. You can *not* have one without the other.

Any time you have an accumulation of such electric charges (say after rubbing your nylon jacket over your dry hair, or after walking across a dry nylon carpet and reaching for that metal door knob, etc.), those charges radiate an electric field out into space. This Electric field manifests itself anytime those accumulated charges are brought near other charges around them, causing the charges around them to move without any direct contact (see our "Cell Phones 101" tutorial on EpiphanyBySteveLee.com, misc. tab for a simple experiment demonstrating this "action at a distance" effect, as well as a discussion on how we use these ElectroMagnetic fields to "telecommunicate"). By putting these charges in motion (say in a radio antenna), we can use them to send "encoded" radio waves out across great distances (e.g. broadcast FM radio, TV, cell phones, radio signals to NASA space probes, etc.).

Since these fields distribute themselves out into space, we obviously need to be able to describe how these fields are distributed across the 3-dimensions of real space (plus time of course). Hence the "x,y,z, t" part of our field notations listed above.

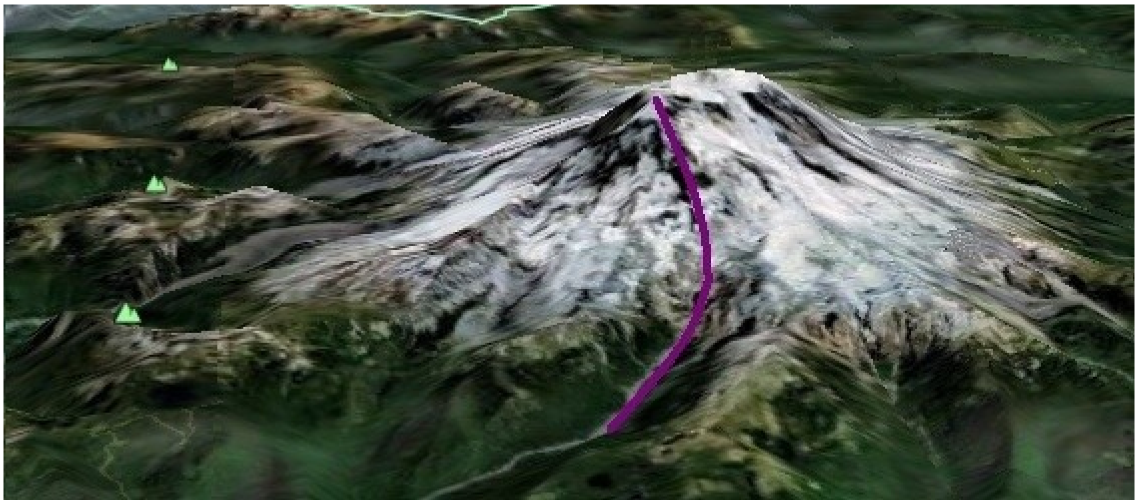
Are you still with me here, or have you run off crying to your mother already?

Okay. Since these fields are 3-dimensional (plus time), in order to understand what these fields are doing, we need some way to analyze their shape and distribution in space and time (e.g. How strong are these fields along each axis? Are they stronger in one direction than another? Can we use that preferred direction to allow us to "aim" them in a given direction and thus focus energy down a given axis? Etc.)

To answer such questions (i.e. to define how a field is distributed throughout space), we need to introduce our next symbol, namely the "Del operator" (no relation to Del Shannon; yeah okay, that would have been mildly amusing in the 70's, but now maybe not so much).

The Del Operator (▼(x,y,z)) effectively finds the 3-dimensional “spatial rate of change” of a field. So when we apply the Del Operator to a field, the result describes how the field distribution and strength changes along each of the three spatial axes (e.g. x,y,z):

For example, if we have a function (“**F**”) that describes a mountain (e.g. Mount Bachelor in Oregon), we can apply the Del Operator to that field to determine the quickest way down that mountain (which of course would be very useful if you had a death wish):



Applying the Del Operator to our fields to help describe the distribution and strength of these fields across some region of space, gives us the “*divergence*” of these fields:

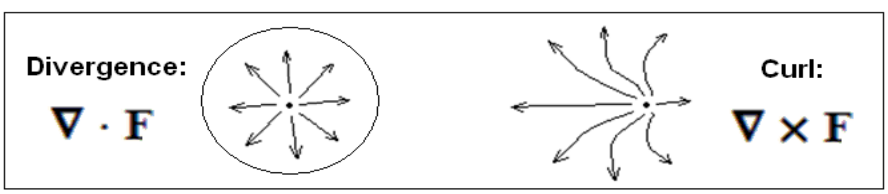
▼ ● **E** (x,y,z, t)

▼ ● **B** (x,y,z, t)

Way back in the 1800's, some guy named Helmhotz realized that in order to completely define any 3-dimensional field, we need to determine two things about that field:

1) its “divergence” = ▼ ● **F** (the total energy radiating out from the field's source), and:

2) its “curl” = ▼ **x F** (also known as the field's “rotation”, i.e. how that field rotates or curls as it distributes itself out into the space around it).



So if we define two more symbols: “q” as our collection of charges, and “ρ” as the charge density (i.e. the amount of “q” per unit of volume), then we can relate the *divergence* of our Electric Field to its source, “ρ”, using what is known as “*Gauss's Law*”:

Gauss's Law: If we construct an empty “shell” around a net charge density and measure the total Electric Field energy hitting that shell from the net charge, Gauss's Law tells us that the total field divergence/energy we measure hitting that shell will be equal to the total field energy that net charge is radiating:

Gauss' Law: $$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

Where "ε" is the "permittivity" of the space around the charge (i.e. how the molecules in this space affect the Electric Field, since we know that different materials attenuate or otherwise modify the Electric Field as it radiates through the material).

If you think about it, Gauss's Law should make perfect sense in terms of Conservation of Energy, since Gauss's Law tells us that the total amount of Electric Field energy measured around the field's source ("ρ"), is equal to the total energy radiated out by that source.

Gauss's Law applied to the Magnetic field looks very similar, except that in this case, the net magnetic charge is zero. It is zero since there is no such thing as an isolated magnetic monopole charge (i.e. all magnets, no matter how small have both poles, a North and a South pole – even when we cut one in half in an attempt to isolate one pole, the resulting two pieces still end up having both a North and a South pole). As a result, Gauss's Law applied to the Magnetic field equals zero:

Gauss' Law: $$\nabla \cdot \mathbf{B} = \cancel{\mu \rho_B} = 0$$

Where, "μ" is the "susceptibility" of the space around the charge (i.e. how the molecules in the area affect the Magnetic Field).

Congratulations, you have now made it *half* way through Maxwell's four equations. To talk about the other two equations (which deal with the curl of the fields as the other field changes in time), we need to first introduce another symbol which we will call the "time-change" operator, "∂F/∂t". This operator is very similar to the Del Operator, except that it is used to determine how much something changes over *time* (rather than over *distance*).

So with that we ask, if the Magnetic Field changes over time, what does the Electric Field do? Faraday's Law tells us that anytime the Magnetic Field changes over time, we see a corresponding curl in the Electric Field:

Faraday's Law: $$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} - \cancel{\mu \mathbf{J}_B}$$

With "J_B" being the magnetic charge current density.

Two things to note about this equation: 1) since there are no magnetic monopole charges, the magnetic charge current density "J_B" = 0 (we included it here so you can compare it to the next equation). Also, 2) there is a negative sign in front of the "time operator". Faraday's Law tells us that anytime a Magnetic Field changes in time, it creates a curled Electric Field with the *opposite polarity* (compared to the magnetic field). Some literature refers to this opposing curled Electric Field, as a "counter Electro-Magnetic Force" (EMF), I.e. an induced voltage that opposes the creation of the Magnetic Field. The importance of the negative sign will be seen in our example below.

Finally, we ask what happens to the Magnetic Field anytime the Electric Field changes in time? Based on symmetry, we suspect a curled Magnetic Field to be created. In fact Ampre's Law tells us that is exactly what happens:

Ampre's Law: $$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} + \mu \mathbf{J}$$

Where "c" is the speed of light, and "J" is the flow of electric charges (i.e. an electric charge current). Another way of stating Ampre's Law is to say that anytime we have a charge flow (J) –or– we have a change in time of the Electric Field, *either* event creates a curled Magnetic Field.

With all four equations together, we see that the first two and the last two are almost symmetrical (mirror images of each other). The lack of the magnetic monopole is all that denies us that perfect mirrored symmetry:

Gauss' Law: $$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

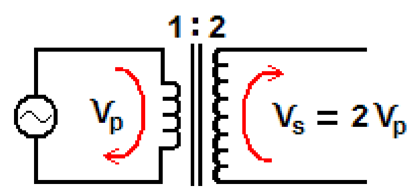
Gauss' Law: $$\nabla \cdot \mathbf{B} = 0$$

Faraday's Law: $$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

Ampre's Law: $$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} + \mu \mathbf{J}$$

To demonstrate the kind of insights offered by Maxwell's four *innocent* looking little equations, we offer the following examples:

<u>Example 1: power transformers</u>:



From Ampre's Law, we see that if we have a time-varying current flow (J) flowing in the primary side of a transformer, we create a curled Magnetic Field that consequently is changing in time with the current flow J. As this changing Magnetic Field couples into the secondary side of the transformer, Faraday's Law tells us that anytime we create such a time-varying Magnetic Field, it creates a counter "EMF" (or an induced voltage) of opposite polarity in the secondary side. If we have twice the amount of wire in the secondary as in the primary (i.e. twice the number of windings in the secondary), we have twice the number of electrons exposed to that time-varying field, creating twice the induced voltage in the secondary (i.e. a step-up transformer).

The negative sign in Faraday's Law tells us that the induced voltage has an opposite polarity and thus opposes the time varying Magnetic Field. With a little thought, this too should make sense: If the induced voltage had the *same* polarity, it would *increase* the Magnetic Field that created it. This increased Magnetic Field would then induce even more voltage, which would then further increase the Magnetic Field, etc., leading very quickly to a run-away explosion of energy. Since this is not what we see in practice, but instead see a measurable opposition to current flow, the negative sign is correct.

Example 2: Kirchoff vs. Faraday:

Let us first rewrite Faraday's Law as:

$$\oint E \bullet dL \;=\; -\int \partial_t B \bullet dA$$

(with § meaning "sum around the closed loop").

Those who have worked with electronic circuits are no doubt familiar with Kirchoff's Voltage Law, which states: "The sum of the voltage drops around a circuit is zero" (note that this is also an expression of the Conservation of Energy, in that it tells us that the voltage dropped by all the loads in a circuit equal the total voltage applied by the battery/generator).

However anyone who has worked near a strong transmitter (e.g. radio broadcast tower, etc.) knows from personal experience that the battery or generator connected to the circuit is *not* the only source of energy entering into that circuit. Truth is, Kirchoff's Law is only a *special case* of Faraday's law, valid only when there is no intruding magnetic field (i.e. **B** = 0):

$$\oint E \bullet dL = -\int \cancel{\partial_t B} \bullet dA = 0$$

Example 3: deriving the Wave Equation:

So now we are in a position to ask "why do we describe the Electric and Magnetic Fields propagating as waves"? Because that's what Maxwell Equations tell us:

Start by taking the curl of Faraday's law, and then use a substitution involving what is know as the "BAC – CAB" identity:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} \nabla \times \mathbf{B}$$

using identity on Left side:
A x B x C = B A C – C A B

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \frac{-1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} - \mu \frac{\partial}{\partial t} \mathbf{J}$$

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} - \nabla^2 \mathbf{E} = -\nabla\left(\frac{\rho}{\varepsilon}\right) - \mu \frac{\partial}{\partial t} \mathbf{J}$$

$$\boxed{\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} - \nabla^2 \mathbf{E} = 0}$$

Freespace Wave Eqn

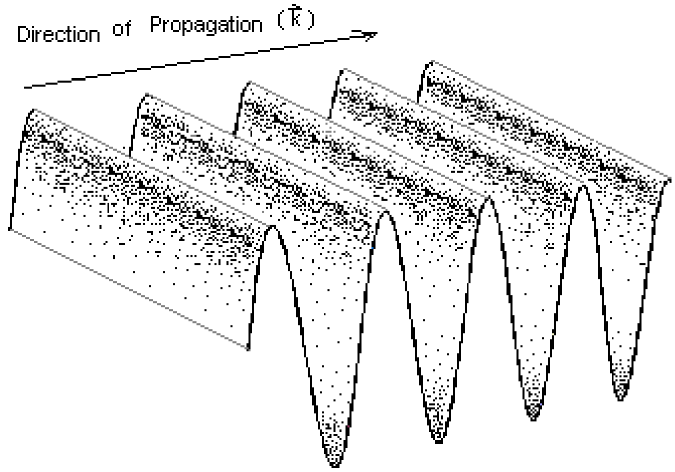
In free space (i.e. no charges "ρ", or currents "J"), the right side therefore goes to zero. This result tells us that the time change of the Electric Field equals the spatial change of that field. This suggests the **E** is an "undulating" (wave-type) function, such as:

$$\mathbf{E} = \mathbf{E_0}\, \mathbf{Sin(aX\ t)} \cdot \mathbf{Sin(aY\ t)} \cdot \mathbf{Sin(aZ\ t)}$$

Where **Eo** is the absolute magnitude of the wave.

If the wave were restricted to propagate solely along a single axis (i.e. aimed in a specific direction), that wave might look something like the following:

$$E = E_o \, Sin(aX\, t)$$



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And that's it, Maxwell's Equations in a nutshell. And you did it without tripping over your skirt even once. As your 1st grade *drill sergeant* of a teacher might say: "*Boo-yah*! Staple a gold star right in the center of that giant melon head of yours!"